$$\cot x=\frac{1}{x}-\frac{1}{\pi-x}+\frac{1}{\pi+x}-\frac{1}{2\pi-x}+\frac{1}{2\pi+x}-\ldots^{*}$$

$$=\cfrac{1}{x+\cfrac{x^2}{\pi-2x+\cfrac{(\pi-x)^2}{2x+\cfrac{(\pi+x)^2}{(\pi-2x)+\ldots}}}}$$

$$\sec x=\frac{4\pi}{\pi^2-4x^2}-\frac{3.4\pi}{3^2\pi^2-4x^2}+\frac{4.5\pi}{5^2\pi^2-4x^2}-\frac{4.7\pi}{7^2\pi^2-4x^2}+\frac{4.9\pi}{9^2\pi^2-4x^2}-\ldots\dagger$$

$$=\cfrac{4\pi}{\pi^2-4x^2+\cfrac{3(\pi^2-4x^2)^2}{6\pi^2+8x^2+\cfrac{(9\pi^2-4x^2)^2}{30\pi^2+8x^2+\ldots}}}$$

$$\csc x=\frac{1}{x}+\frac{2x}{\pi^2-x^2}-\frac{2x}{2^2\pi^2-x^2}+\frac{2x}{3^2\pi^2-x^2}-\frac{2x}{4^2\pi^2-x^2}+\ldots$$

$$=a-\frac{ax-1}{x}+\frac{2x}{\pi^2-x^2}-\frac{2x}{4\pi^2-x^2}+\ldots\ddagger$$

$$=\cfrac{a}{1+\cfrac{ax-1}{1+\cfrac{2ax^3}{a\pi^2x-\pi^2-x^2-ax^3+\cfrac{2x(ax-2)(\pi^2-x^2)^2}{6\pi^2+\ldots}}}}$$

Also solved by G. W. Greenwood.

268. Proposed by O. E. GLENN, Ph. D., Philadelphia, Pa.

Express the hyperbolic functions of x in the form of infinite continued fractions.

Solution by J. SCHEFFER, A. M., Professor of Mathematics, Kee Mar College, Hagerstown, Md.

For the conversion of $\tan x$ and $\tanh x$, Legendre has given a very elegant solution.

Putting $f(x)=1+\dfrac{a}{x}+\frac{1}{2}.\dfrac{a^2}{x(x+1)}+\dfrac{1}{2.3}.\dfrac{a^3}{x(x+1)(x+2)}+\ldots$ we have

$$f(x+1)=1+\frac{a}{x+1}+\tfrac{1}{2}.\frac{a^2}{(x+1)(x+2)}+\frac{1}{2.3}.\frac{a^3}{(x+1)(x+2)(x+3)}+\ldots$$

$$\therefore f(x)-f(x+1)=\frac{a}{x(x+1)}+\frac{a^2}{x(x+1)(x+2)}+\tfrac{1}{2}\frac{a^3}{x(x+1)(x+2)(x+3)}+\ldots$$

**Encyclopedia Britannica*, Vol. XXIII, p. 572.

†Euler's *Institu. Calc. Diff.*, II, 8, No. 223—224.

‡The quantity a may have any integral value which will avoid negative terms in the continued fraction.

$$=\frac{a}{x(x+1)}\left[1+\frac{a}{x+2}+\frac{1}{2}\frac{a^2}{(x+2)(x+3)}+\dots\right]=\frac{a}{x\,(x+1)}f(x+2)\,;$$

whence, putting $F(x)=\frac{a}{x}.\frac{f(x+1)}{f(x)}$, $F(x+1)=\frac{a}{x+1}\,\frac{f(x+2)}{f(x+1)}$, we obtain

$$\frac{x}{a}\frac{f(x)}{f(x+1)}-\frac{x}{a}=\frac{1}{x+1}.\frac{f(x+2)}{f(x+1)}.\quad \frac{1}{F(x)}-\frac{x}{a}=\frac{F(x+1)}{a},\quad F(x)=\frac{a}{x+F(x+1)}$$

$$\therefore F(x+1)=\frac{a}{x+1+F(x+1)},\quad F(x+2)=\frac{a}{x+2+F(x+2)},\ \text{etc.}$$

$$\therefore F(x)=\cfrac{a}{x+\cfrac{a}{x+1+\cfrac{a}{x+2+\cfrac{a}{x+3+\dots}}}},$$

where $F(x)=\dfrac{a}{x}\,\dfrac{1+\frac{a}{x+1}+\frac{1}{2}\frac{a^2}{(x+1)(x+2)}+\frac{1}{2.3}\frac{a^3}{(x+1)(x+2)(x+3)}+\dots}{1+\frac{a}{x}+\frac{1}{2}\frac{a^2}{x(x+1)}+\frac{1}{2.3}\frac{a^3}{x(x+1)(x+2)}+\dots}$

For $x=\frac{1}{2}$, $F(x)=2a.\dfrac{1+\frac{4a}{2.3}+\frac{16a^2}{2.3.4.5}+\frac{64a^3}{2.3.4.5.6.7}+\dots}{1+\frac{4a}{2}+\frac{16a^2}{2.3.4}+\frac{64a^3}{2.3.4.5.6}+\dots}=\dfrac{\epsilon^{2\sqrt{a}}-\epsilon^{-2\sqrt{a}}}{\epsilon^{2\sqrt{a}}+\epsilon^{-2\sqrt{a}}}\sqrt{a}.$

$$\therefore \frac{e^{2\sqrt{a}}-e^{-2\sqrt{a}}}{e^{2\sqrt{a}}+e^{-2\sqrt{a}}}2\sqrt{a}=\cfrac{4a}{1+\cfrac{4a}{4+\cfrac{4a}{5+\dots}}}$$

Putting $2\sqrt{a}=x$, $\dfrac{e^x-e^{-x}}{e^x+e^{-x}}=\tanh x=\cfrac{x}{1+\cfrac{x^2}{3+\cfrac{x^2}{5+\cfrac{x^3}{7+\dots}}}}$

Also $\dfrac{e^{ix}-e^{-ix}}{e^{ix}+e^{-ix}}=\cfrac{ix}{1-\cfrac{x^2}{3-\cfrac{x^2}{5-\dots}}}$

$$\therefore i\tan x=\cfrac{ix}{1-\cfrac{x^2}{3-\cfrac{x^2}{5-\cfrac{ix^3}{7-\dots}}}}\qquad \text{or, } \tan x=\cfrac{x}{1-\cfrac{x^2}{3-\cfrac{x^2}{5-\dots}}}$$

a very simple and remarkable continued fraction.

Since $\sinh x = -i\sin ix$, and $\cosh x = \cos ix$, we get, by substituting ix for x in the continued fractions for $\sin x$ and $\cos x$,

$$\sinh x = \frac{x}{1} - \frac{x^3}{2.3+x^2-} \frac{2.3x^2}{4.5+x^2-} \frac{4.5x^2}{6.7+x^2-\text{ etc.,}}$$

$$\cosh x = \frac{1}{1} - \frac{1}{2+x^2-} \frac{2x^2}{3.4+x^2-} \frac{3.4x^2}{5.6+x^2-\text{ etc.}}$$

GEOMETRY.

289 (Incorrectly numbered 288). Proposed by C. N. SCHMALL, College of the City of New York.

From a point P on a given circle to draw two chords such that, (α) chord PA : chord $PB = m : n$ (a given ratio), and, (β) arc PA : arc $PB = 1 : 3$.

Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

Let O be the center of the circle, radius r. Also let angle $POA = 2\theta$, angle $POB = 6\theta$. Then $PA = 2r\sin\theta$, $PB = 2r\sin3\theta$.

$$\frac{PA}{PB} = \frac{\sin\theta}{\sin3\theta} = \frac{1}{3-4\sin^2\theta} = \frac{m}{n}. \quad \therefore \sin\theta = \tfrac{1}{2}\sqrt{\frac{3m-n}{m}},\ \sin3\theta = \frac{n}{2m}\sqrt{\frac{3m-n}{m}}.$$

$$\therefore \text{Make angle } POA = 2\sin^{-1}\tfrac{1}{2}\sqrt{\frac{3m-n}{m}};\ \text{angle } POB = 2\sin^{-1}\frac{n}{2m}\sqrt{\frac{3m-n}{m}}.$$

Then chord PA : chord $PB = m : n$; arc PA : arc $PB = 1 : 3$.

290 (Incorrectly numbered 289). Proposed by J. J. QUINN, Ph. D., Scottdale, Pa.

(a) Suppose a circle described around the origin. Then at the end of a uniformly revolving radius r, a line equal to the diameter is pivoted. Find the equation of the locus of its extremity, if for every unit of angle its projection on the X axis is a constant linear unit, being the same part of the diameter as the angle is of π radians.

(b) Show how it can be applied to the trisection or multisection of an angle.

No solution has been received.

292 (Incorrectly numbered 290). Proposed by DR. L. E. DICKSON, The University of Chicago, Chicago, Ill.

Given nine points lying by threes in three columns and in three rows, draw through them, by continuous motion, a broken line composed of only four straight segments, and passing but once through each of the nine points. [A current puzzle.]